Numerical Analysis

Lecture 22

Chapter 5 Interpolation

Finite Difference Operators
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Lagrange’s Interpolation Formula
Divided Differences
Interpolation in Two Dimensions

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1},$$

$$i = n, (n-1), \ldots, k$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

Thus

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

Shift operator, E

$$E\,f(x) = f(x+h)$$

$$E^n f(x) = f(x+nh)$$

$$E^n y_x = y_{x+nh}$$

The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x - h)$$

Similarly,

$$E^{-n} f(x) = f(x - nh)$$

Average Operator μ

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2} \right) + f\left(x - \frac{h}{2} \right) \right]$$

$$= \frac{1}{2}\left[y_{x+(h/2)} + y_{x-(h/2)} \right]$$

Differential Operator, *D*

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f^{n}(x)$$

Important Results

$$\Delta = E - 1$$

$$\nabla = 1 - E^{-1} = \frac{E-1}{E}$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$hD = \log E$$

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

Newton's Forward Difference Interpolation Formula

Let $y = f(x)$ be a function which takes values $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, ..., corresponding to various equi-spaced values of x with spacing h, say x_0, $x_0 + h$, $x_0 + 2h$, ...

Suppose, we wish to evaluate the function $f(x)$ for a value $x_0 + ph$, where p is any real number, then for any real

$$E^p f(x) = f(x + ph).$$

$$f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p f(x_0)$$

$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots \right] f(x_0)$$

$$f(x_0+ph)=f(x_0)+p\Delta f(x_0)$$
$$+\frac{p(p-1)}{2!}\Delta^2 f(x_0)+\frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0)$$
$$+\cdots+\frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n f(x_0)+\text{Error}$$

This is known as Newton's forward difference formula for interpolation, which gives the value of $f(x_0+ph)$ in terms of $f(x_0)$ and its leading differences.

This formula is also known as Newton-Gregory forward difference interpolation formula. Here p=(x-x_0)/h.

An alternate expression is

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error}$$

Exercise

Find a cubic polynomial in *x* which takes on the values -3, 3, 11, 27, 57 and 107, when *x* = 0, 1, 2, 3, 4 and 5 respectively.

Solution

Here, the observations are given at equal intervals of unit width.

To determine the required polynomial, we first construct the difference table

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3	6		
1	3	8	2	
2	11	16	8	6
3	27	30	14	6
4	57	30	20	6
5	107	50		

Since the 4th and higher order differences are zero, the required Newton's interpolation

$$f(x_0+ph)=f(x_0)+p\Delta f(x_0)+\frac{p(p-1)}{2}\Delta^2 f(x_0)$$

$$+\frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0)$$

Here,

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

$$\Delta f(x_0) = 6$$

$$\Delta^2 f(x_0) = 2$$

$$\Delta^3 f(x_0) = 6$$

Substituting these values into the formula, w

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

$$f(x) = x^3 - 2x^2 + 7x - 3,$$

The required cubic polynomial.

NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

For interpolating the value of the function $y = f(x)$ near the end of table of values, and to extrapolate value of the function a short distance forward from y_n, Newton's backward interpolation formula is

Derivation

Let $y = f(x)$ be a function which takes on values $f(x_n)$, $f(x_n\text{-h})$, $f(x_n\text{-2h})$, ..., $f(x_0)$ corresponding to equispaced values x_n, x_n-h, x_n-2h, ..., x_0. Suppose, we wish to evaluate the function $f(x)$ at $(x_n +$

where *p* is any real number, then we have the shift operator *E*, such that

$$f(x_n + ph) = E^p f(x_n) = (E^{-1})^{-p} f(x_n) = (1 - \nabla)^{-p} f(x_n)$$

Binomial expansion yields,

$$f(x_n + ph) = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n + \text{Error} \right] f(x_n)$$

That is,

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n)$$

$$+ \frac{p(p+1)}{2!}\nabla^2 f(x_n)$$

$$+ \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots$$

$$+ \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n f(x_n) + \text{Error}$$

This formula is known as Newton’s backward interpolation formula. This formula is also known as Newton’s-Gregory backward difference interpolation formula.

If we retain $(r + 1)$ terms, we obtain a polynomial of degree r agreeing with $f(x)$ at x_n, x_{n-1}, ..., x_{n-r}. Alternatively, this f ... n as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error}$$

Here $p = \dfrac{x - x_n}{h}$

Example

For the following table of values, estimate *f* (7.5).

x	1	2	3	4	5	6	7	8
$y = f(x)$	1	8	27	64	125	216	343	512

Solution

The value to be interpolated is at the end of the table. Hence, it is appropriate to use Newton's backward interpolation formula. Let us first construct the backward difference table

Difference Table

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
2	8	7			
3	27	19	12		
4	64	37	18	6	
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0

Since the 4th and higher order differences are zero, the required Newton's backward interpolation f

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$$

$$+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

In this problem,

$$p = \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5$$

$$\nabla y_n = 169, \quad \nabla^2 y_n = 42, \quad \nabla^3 y_n = 6$$

$$y_{7.5} = 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2}(42)$$

$$+ \frac{(-0.5)(0.5)(1.5)}{6}(6)$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875$$

Example

The sales for the last five years is given in the table below. Estimate the sales for the year 1979

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Solution

Newton's backward difference table for the given data as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
1976	43	3			
1978	48	5	2		
1980	52	4	-1	**-3**	
1982	57	5	1	2	5

In this example,

$$p = \frac{1979 - 1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \ \nabla^2 y_n = 1,$$

$$\nabla^3 y_n = 2, \ \nabla^4 y_n = 5$$

Newton's interpolation formula gives

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1)$$

$$+ \frac{(-1.5)(-0.5)(0.5)}{6}(2)$$

$$+ \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

Therefore, $y_{1979} = 50.1172$

LAGRANGE’S INTERPOLATIO N FORMULA

Newton's interpolation formulae developed earlier can be used only when the values of the independent variable *x* are equally spaced. Also the differences of *y* must ultimately become

If the values of the independent variable are not given at equidistant intervals, then we have the basic formula associated with the name of Lagrange which will be derived now.

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function which takes the values, $y_0, y_1, \ldots y_n$ corresponding to $x_0, x_1, \ldots x_n$. Since there are (n + 1) values of y corresponding to (n + 1) values of x, we can represent the function f (x) by a polynomial of degree n.

Suppose we write this polynomial in the form .

$$f(x) = A_0 x^n + A_1 x^{n-1} + \cdots + A_n$$

or in the form

$$\begin{aligned} y = f(x) &= a_0 (x - x_1)(x - x_2)\cdots(x - x_n) \\ &+ a_1 (x - x_0)(x - x_2)\cdots(x - x_n) \\ &+ a_2 (x - x_0)(x - x_1)\cdots(x - x_n) + \cdots \\ &+ a_n (x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

Here, the coefficients a_k are so chosen as to satisfy this equation by the (n + 1) pairs (x_i, y_i).

Thus we get

$$y_0 = f(x_0) = a_0(x_0 - x_1)(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)$$

Therefore,

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}$$

Similarly, we obtain

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}$$

$$a_i = \frac{y_i}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

and

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, \ldots, a_n$ we get

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}y_i + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}y_n$$

The Lagrange's formula for interpolation

This formula can be used whether the values x_0, x_2, …, x_n are equally spaced or not.

Alternatively, this can also be written in compact fo

$$y = f(x) = L_0(x)\,y_0 + L_1(x_1)\,y_1 + L_i(x_i)\,y_i + \cdots + L_n(x_n)\,y_n$$

$$= \sum_{k=0}^{n} L_k(x)\,y_k \qquad = \sum_{k=0}^{n} L_k(x)\,f(x_k)$$

Where,

$$L_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

We can easily observe that,

$L_i(x_i) = 1$ **and** $L_i(x_j) = 0, i \neq j.$

Thus introducing *Kronecker* delta notation

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Further, if we introduce the notation

$$\Pi(x) = \prod_{i=0}^{n}(x - x_i) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

That is $\Pi(x)$ is a product of (n + 1) factors. Clearly, its derivative $\Pi'(x)$ contains a sum of (n + 1) terms in each of which one of the factors of will $\Pi(x)$ be absent.

We also define,

$$P_k(x) = \prod_{i \neq k} (x - x_i)$$

which is same as $\prod(x)$ except that the factor $(x - x_k)$ is absent. Then

$$\prod{}'(x) = P_0(x) + P_1(x) + \cdots + P_n(x)$$

But, when $x = x_k$, all terms in the above sum vanishes except $P_k(x_k)$

Hence,

$$\Pi'(x_k) = P_k(x_k) = (x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)$$

$$L_k(x) = \frac{P_k(x)}{P_k(x_k)} = \frac{P_k(x)}{\Pi'(x_k)}$$

$$= \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)}$$

Finally, the Lagrange's interpolation polynomial of degree n can be written as

$$y(x) = f(x) = \sum_{k=0}^{n} \frac{\prod(x)}{(x - x_k)\prod'(x_k)} f(x_k)$$

$$= \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} L_k(x) y_k$$

Example

Find Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$. Hence find $y(5)$.

Solution

The given data can be arranged as

x	1	3	4	6
$y = f(x)$	-3	0	30	132

Using Lagrange's interpolation formula, we have

$$y(x) = f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3)$$

$$+\frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$

$$+\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30)$$

$$+\frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}\left(-5x^3 + 135x^2 - 460x + 300\right)$$

$$= \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

which is required Lagrange's interpolation polynomial.

Now, $y(5) = 75$.

Example

Given the following data, evaluate *f* (3) using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution

Using Lagrange's formula,

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+$$

$$\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Therefore

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1)$$

$$+\frac{(3-1)(3-5)}{(2-1)(2-5)}(4)$$

$$+\frac{(3-1)(3-2)}{(5-1)(5-2)}(10)$$

$$= 6.49999 = 6.5$$